OPTIMAL EXCHANGE BETTING STRATEGY FOR WIN-DRAW-LOSS MARKETS

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Abstract

Since the Betfair betting exchange launched in 2000, sports gamblers have had a gambling forum quite different from the traditional bookmaker. Three features of betting exchanges in particular require new analysis methods extending the Kelly criterion originated by John Kelly (1956):

(i) The ability to lay (i.e., bet against) a team as well as back it
(ii) Negotiation of odds, where one can set one's own odds and wait for another punter to match them, not just accept the market valuation at the time
(iii) The bookmaker takes a fee as a fixed fraction of one's net profit on a market, not as a hidden margin in each betting option's price

In sports where there are more than two possible outcomes, such as soccer (football), usually the prospective gambler will find that if he/she wants to bet on one team using the Kelly criterion, the same criterion will advocate laying against the other team. Basic Kelly betting offers no resolution to these correlated markets, and some punters at traditional bookmakers will instead seek a binary 'handicap' or 'draw-no-bet' market in order to find prices that they can immediately understand.

This paper derives the criterion one should use when investing in a 'win-draw-loss' market, with the important feature that profits are significantly higher by combining back and lay bets than by relying on one or the other. The 'draw-no-bet' approach is shown to be optimal only in a narrow band of cases, where the advantage of having the draw result untaxed outweighs the profits to be gained by effectively backing it.

Keywords: Betting exchange, Kelly criterion, Betfair, Football, Soccer

1. INTRODUCTION

The Kelly criterion (Kelly, 1956) is a vital tool in the armoury of both portfolio investors and gamblers. By maximising logarithmic utility – simultaneously minimising the risk of ruin – Kelly provided the formula that gamblers with perfect probabilistic knowledge must use to grow their bank at the largest expected rate.

With the explosion in sports betting around the world since the rise of the internet, several papers have been written expanding the Kelly criterion, for instance to account for multiple simultaneous independent market investments (Thorp, 1997; Insley et al, 2004), spread betting (Chapman, 2007), and back/lay comparison on betting exchanges (Walshaw, 2010). Barnett (2011) applied the Kelly criterion to the game of Video Poker, where there are multiple possible outcomes but only a single bet to make on each hand.

Meanwhile, betting exchanges have provided markets that are often more attractive than regular bookmakers, by allowing punters to match their money with peers. Soccer (association football) matches are some of the most popular and therefore most liquid markets, with over \$100 million matched on the final of the 2010 FIFA World Cup (Betfair, 2010).

This paper addresses dilemmas that gamblers can feel in sports such as soccer and cricket that have a three-option market. Is it more profitable to back the team that the punter believes is underrated by the market, or match someone else's money by laying the team that appears overrated? And is the "draw no bet" market worthwhile?

2. METHODS

The basic Kelly criterion for a single option on a regular betting market gives the Kelly Bet B as:

$$B \;=\; \frac{Mp-1}{M-1} \qquad (1)$$

where M is the team's market price and p is the gambler's presumed probability of the team winning. B is expressed as a percentage of the bettor's bankroll, and a bet should be placed if $Mp>1$. The formula is derived by maximising *log*(expected bank) with respect to the bet proportion B.

Betfair, which comprises about 90% of the betting exchange economy worldwide (Sydney Morning Herald, 2006), does not build its profit margin into each price like a traditional bookmaker, but instead 'taxes' each market winner on their net profit once the event is resolved. A gambler could have several individual bets on the same market, even arbitraging a guaranteed profit as the odds change, and only pay a fee on his/her net result on the winning option(s). The level of tax t varies from 5% for low-volume gamblers down to 2% for those who have the largest betting history. This leads to an adjusted formula, where M_B is the agreed price for the bet:

$$B \;=\; \frac{(M_B-1)(1-t)p-(1-p)}{(M_B-1)(1-t)} \qquad (2)$$

It is immediately obvious that for a single bet, taking Betfair odds M_B is exactly equivalent to taking a slightly lower price at a standard betting shop:

$$M' \;=\; 1+(M_B-1)(1-t) \qquad (3)$$

e.g. for a 5% tax, Betfair \$2 is equivalent to traditional \$1.95 while Betfair \$1.20 is equivalent to \$1.19 at a regular bookmaker.

In lay betting, the punter risks $L(M_L-1)$ by accepting a bet of size L from an anonymous peer, having negotiated a price M_L. The Kelly Bet in this case is:

$$L \;=\; \frac{(M_L-1)(1-t)(1-p)-p}{(1-t)} \qquad (4)$$

In this paper, we limit our analysis to a combination of betting on one team and laying against the other, ignoring the market price for the central option (the draw). This is done without loss of generality if the market is fully saturated and has automated bet-matching, which is usually the case for Betfair soccer markets. Under this assumption, the draw price can be derived as $M_D = 1/[1-(1/M_B)-(1/M_L)]$ but the other two prices capture all necessary market information. We also do not consider the risks and benefits of setting our own odds and waiting for the market to match them, although this should be part of a practical application along with an assessment of the reliability of the punter's presumed probabilities.

To find the optimal combination of bets $\{B,L\}$ we must go back to first principles and maximise W, the log of the expected bankroll. At a traditional bookmaker who offers the equivalent of 'lay' odds (usually called a 'second chance' or 'win or draw' market), this is easily solved.

$$
\begin{aligned}
W &= p_L \log(1 - B - L(M_L - 1)) \\
&\quad + (1 - p_L - p_B)\log(1 - B + L) \\
&\quad + p_B \log(1 + B(M_B - 1) + L) \qquad (5)
\end{aligned}
$$

Solving for

$$\frac{\partial W}{\partial B} = 0 \text{ and } \frac{\partial W}{\partial L} = 0 \qquad (6)$$

We find that the maximum bankroll growth is achieved when

$$B_0 = \frac{M_B p_B (M_L - 1) - M_L (1 - p_L)}{(M_B - 1)(M_L - 1) - 1}$$

and (7)

$$L_0 = \frac{M_B (1 - p_B) - M_L p_L (M_B - 1)}{(M_B - 1)(M_L - 1) - 1}$$

provided both B_0 and L_0 are positive. If only one is positive, the punter should revert to (1).

The situation with tax is more complicated, requiring maximising of the function:

$$
\begin{aligned}
W_t &= p_L \log(1 - B - L(M_L - 1)) \\
&\quad + (1 - p_L - p_B)\log(1 + (1 - t)^{\mathrm{H}(L-B)}(L - B)) \\
&\quad + p_B \log(1 + (1 - t)(B(M_B - 1) + L)) \qquad (8)
\end{aligned}
$$

where $\mathrm{H}(x)$ is the Heaviside step function, indicating that the draw result is only taxed if the lay is larger than the bet.

3. RESULTS

First, consider the simpler situation described in equation (7) where a traditional bookmaker offers odds to bet on a win-or-draw market. In general, these prices tend to be unattractive as the bookmaker has built a substantial profit margin for itself into them; however they are a useful demonstration of the method for the more complex betting exchange situation.

In this example, City is playing United. Our hypothetical punter with a \$1,000 bankroll believes that the true probabilities are 60% City wins, 15% United wins, and 25% the match will be drawn. The bookmaker is offering M_B = \$2.00 about City, and a "draw or City" market at \$1.30, equivalent to laying United at M_L = \$4.33.

Using the independent formula (1), the Kelly criterion would advise us to bet 20% or \$200 on City. The mean profit is expected to be (\$400 × 60% – \$200) = \$40.

Considering instead the "draw or City" market, the Kelly criterion advises a bet of size \$350 (35%), equivalent to laying United for \$105 = \$350 × (1.3–1). The mean profit is expected to be (\$455 × 85%– \$350) = \$36.75.

To reconcile these two criteria, we must use formula (7) to find $\{B_0 = 0.13571,\ L_0 = 0.06429\}$. I.e., bet \$135.71 on City and simultaneously bet \$214.29 on "draw or City" (equivalent to laying United for \$64.29). For an exposure of \$350 – in this case the same as "draw or City" alone – the punter has increased his expected profit to \$49.64, or 14.2% of his outlay.

Paradoxically, it should be noted that the punter has effectively taken a price on the draw outcome that the Kelly criterion would advise has an expected loss. The punter believes that the draw is a 25% prospect, but the difference between the odds of \$2 (50%) and \$1.30 (77%) is 27%, meaning that he is paying a premium for including this option in his betting portfolio. Additionally, if the match results in a draw he will still suffer a net loss of \$71.43. However, as the goal is to minimise the risk of long-term ruin, the increased diversification to include the draw is the correct strategy.

The Betfair version of this problem in equation (8) must take into account three different possibilities:

i. $L > B$, and the net profit from a draw will be taxed
ii. $L < B$, so the draw will be a net loser
iii. $L = B$, "draw no bet"

The log formula to be maximised is different in all three cases, so the zeroes in the derivatives must be examined for domain relevance and compared with each other.

3.i. The Case L > B

Solving (6) for W_t in (8) with the taxed draw gives:

$$B_1 = B_0 \frac{M_L - t}{(1 - t) M_L} \qquad (9)$$

$$L_1 = L_0 + \frac{t[M_B p_B - M_L^2 p_L (M_B - 1) + M_L (1 - p_L) + M_B M_L (p_L - p_B)]}{(1 - t) M_L [(M_B - 1)(M_L - 1) - 1]}$$

provided $L_1 > B_1$.

3.ii. The Case L < B

Solving (6) for W_t in (8) with the untaxed draw gives:

$$L_2 = L_0 \frac{(1-t)M_B + t}{(1-t)M_B} \qquad (10)$$

$$B_2 = B_0 + \frac{t[M_B(1-p_B) - M_B M_L(1-p_B) + M_L p_L]}{(1-t)M_B[(M_B-1)(M_L-1)-1]}$$

provided $L_2 < B_2$.

3.iii. The Case L = B ("draw no bet")

By eliminating the draw outcome, (6) is simply solved for the first and third terms of (8) with B set to L:

$$B_3 = L_3 = \frac{(1-t)M_B p_B - M_L p_L}{(1-t)M_B M_L (p_B + p_L)} \qquad (11)$$

This is the common boundary of the other two cases.

Examples

Returning to our City vs United example, consider a market where the exchange prices are M_B = \$2.05 about City, and M_L = \$4.50 for United. For an individual market option, this is equivalent to the standard bookmaker's prices earlier in this section after a t=0.05 tax is factored in.

Checking the three case functions, case ii is consistent and outperforms case iii, which dominates case i along the entire boundary (the maximum of W_t occurs outside of the $L > B$ domain). The formula recommends values $\{B_0 = 0.15837,\ L_0 = 0.04266\}$. I.e., bet \$158.37 on City and simultaneously lay United for \$42.66 (lay exposure \$149.30, total exposure \$307.67). The expected net profit on the market is \$44.02, or 14.3% of his outlay. His loss in the case of a draw is \$115.71. Using the exchange tends to push the successful strategy in the direction of betting on the favourite as opposed to laying against the underdog.

To examine how the optimal back/lay proportion changes with varying odds, a series of price sets was generated from the cumulative normal distribution with a fixed z difference of 0.3 between the market odds and the punter's probabilities. The central 'draw' option was given a width of 0.8 on the z scale to mimic real soccer draw odds in professional leagues. For example, the market centred on z=0 would have City and United both on a price of \$2.90 (equivalent to 34.5% probability, or $z < -0.4$). The punter's belief is that City has a 46.0% chance of winning ($z < -0.1$), compared to 24.2% for United ($z > 0.7$). While this method produces superficially credible sets of odds, a more precise simulation of soccer should use an accepted modelling approach such as that recommended by Dixon (1998).

Optimal Back/Lay From \$1,000 bankroll					
City	Draw	United	Case	Back (B)	Lay (L)
\$50	\$11.70	\$1.12	L>B	\$9	\$498
\$20	\$6.75	\$1.25	L>B	\$21	\$350
\$10	\$4.65	\$1.45	L>B	\$41	\$239
\$7.50	\$4.10	\$1.60	L>B	\$53	\$197
\$6.00	\$3.75	\$1.75	L>B	\$66	\$166
\$5.00	\$3.50	\$1.95	L>B	\$79	\$140
\$4.00	\$3.35	\$2.20	L>B	\$98	\$111
\$3.50	\$3.25	\$2.46	L=B	\$104	\$104
\$3.35	\$3.25	\$2.55	L=B	\$104	\$104
\$3.20	\$3.25	\$2.62	L=B	\$103	\$103
\$2.90	\$3.20	\$2.90	L<B	\$114	\$90
\$2.60	\$3.25	\$3.25	L<B	\$128	\$76
\$2.40	\$3.25	\$3.60	L<B	\$139	\$67
\$2.20	\$3.35	\$4.05	L<B	\$152	\$57
\$2.00	\$3.50	\$4.70	L<B	\$169	\$47
\$1.80	\$3.70	\$5.75	L<B	\$190	\$36
\$1.60	\$4.10	\$7.60	L<B	\$219	\$26
\$1.40	\$5.00	\$11.50	L<B	\$259	\$15
\$1.20	\$7.80	\$26	L<B	\$329	\$6
\$1.10	\$13.50	\$62	L<B	\$391	\$2.20

Table 1: Effective Strategy for a Range of Markets

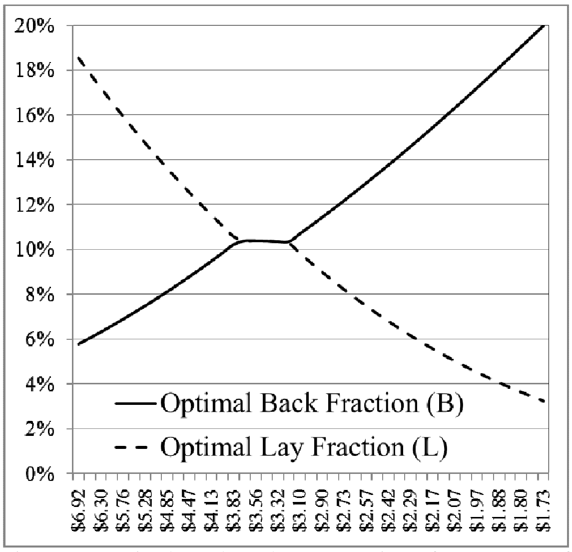

Figure 1: Optimal Back and Lay Fractions for a range of favourite's prices, showing a narrow central zone where 'draw no bet' is the optimal strategy

Table 1 shows the optimal strategy for a range of odds, and Figure 1 plots the function, clearly showing a 'kink' for the narrow range of situations where the gambler should attempt not to make a profit on the draw, in order to avoid tax on that result. The function using standard bookmaker odds does not display such a discontinuity in the derivative.

4. DISCUSSION

Betting via an exchange has subtle and surprising repercussions for optimal gambling strategy. The way in which a bookmaker profits is quite different from the situation at an exchange like Betfair, which does not set the odds centrally but takes a fraction of each payout when the market settles.

This paper has extended the well-known Kelly criterion to the popular sport of soccer, and shown that use of this advanced strategy leads to a substantial increase in expected profit – greater than 20% in our City vs United example.

Naturally, punters in the real world ought to revisit the assumptions of section 2, particularly if the market is not as liquid as they might wish. In practice, blindly using the full Kelly Bet fraction would be a risky rollercoaster for most punters, as they do not account for the error in the probabilities generated when they frame the market. A more complete approach might use a Bayesian distribution of predicted market outcomes, taking into account a variety of known and unknown factors in the game then optimising a more complex log-utility function. Some gamblers use a 'fractional Kelly' system, which assigns an active bankroll to the Kelly formula that is only a fraction of the full bankroll. This works as an approximation of the effect of having limited information, while leaving funds available for betting in other markets simultaneously. It is also wise to wager on the conservative side of the Kelly fraction as the penalty function for overbetting is steep.

The equations derived here provide a handy rule of thumb: in general, a punter who wants to back the favourite should put most of his/her money into that option, while backing the underdog is usually not as efficient in growing the bankroll as laying against the favourite – particularly when the favourite's odds are close to 50/50.

5. CONCLUSIONS

This paper has solved the Kelly criterion for the case of win-draw-loss markets. This is immediately applicable to soccer, cricket, and other sports with a reasonable likelihood of neither team winning such as hockey and chess.

It is also applicable to sports such as Australian Rules Football, where the punter must decide how to allocate his funds to a head-to-head (win/loss) or 'line' bet, which are the two most liquid markets. The range of outcomes between zero and the published line handicap can be treated as the middle outcome in the formulas published here.

More generally, future work could extend the methodology to n published lines and numerically find the optimal W_t for a betting portfolio that would potentially be spread across a number of them.

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